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Periodic function. A function $f(x)$ defined on a given domain is called of period T if

$$f(x+T) = f(x) = f(x+2T) = f(x+3T) = \cdots$$

Example. Let

$$f(x) = \cos x$$

then $f(x+2\pi) = \cos(x+2\pi) = \cos x$

$f(x+4\pi) = \cos(x+4\pi) = \cos x$

etc

$\therefore$ $T = 2\pi$ here

$\therefore$ $\cos x$ is a periodic function of period 2π.

Also $\sin x$ is a periodic function of period 2π as

$$\sin(x+2\pi) = \sin x = \sin(x+4\pi) = \cdots$$

$\tan x$ is a periodic function of period π

as $\tan(x+\pi) = \tan x = \sin(x+2\pi) = \cdots$

Even function. A function $f(x)$ defined on a given interval is called an even function if

$$f(-x) = f(x), \ \forall\, x \in \text{given interval}$$

ex $f(x) = \cos x$, $x \in]-\pi, \pi[$

then $f(x)$ is an even function as $f(-x) = f(x)$

i.e. $\cos(-x) = \cos x$. $\forall\, x \in]-\pi, \pi[$.

Odd function. A function $f(x)$ on a given interval is called an odd function if $f(-x) = -f(x)$, $\forall\, x \in$ given interval.

example. $f(x) = \sin x$ is an odd function as $f(-x) = -f(x)$, $\forall\, x \in$ given interval.

We must remember some formulae as under:

(1) $\int_0^{2\pi} \sin nx = 0$ (2) $\int_0^{2\pi} \cos nx \, dx = 0$

(3) $\int_{-\pi}^{\pi} \sin nx \, dx = 0$

(4) $$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}\left[a \sin bx - b \cos bx\right]$$

(5) $$\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}\left[a \cos bx + b \sin bx\right]$$

Proof of (4) Let

$$I = \int \underset{\text{II}}{e^{ax}} \underset{\text{I}}{\sin bx} \, dx$$

$$I = \sin bx \cdot \frac{e^{ax}}{a} - \int b \cos bx \cdot \frac{1}{a} e^{ax} \, dx$$

$$= \frac{1}{a} e^{ax} \sin bx - \frac{b}{a} \int \cos bx \, e^{ax} \, dx$$

$$= \frac{1}{a} e^{ax} \sin bx - \frac{b}{a}\left[\cos bx \cdot \frac{e^{ax}}{a} - \int (-) b \sin bx \cdot \frac{e^{ax}}{a} \, dx\right]$$

$$I = \frac{1}{a} e^{ax} \sin bx - \frac{b}{a^2} e^{ax} \cos bx - \frac{b^2}{a^2} \underbrace{\int \sin bx \, e^{ax} \, dx}_{= I}$$

$$\left(1 + \frac{b^2}{a^2}\right) I = \frac{a}{a^2} e^{ax} \sin bx - \frac{b}{a^2} e^{ax} \cos bx$$

$$\left(\frac{a^2+b^2}{\cancel{a^2}}\right) I = \frac{1}{\cancel{a^2}} e^{ax}\left[a \sin bx - b \cos bx\right]$$

$$I = \frac{e^{ax}}{a^2+b^2}\left[a \sin bx - b \cos bx\right]$$

$$\therefore \int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}\left[a \sin bx - b \cos bx\right]$$

Similarly we can prove (5).

State and prove Euler's formula for Fourier series.

Statement, Let $f(x)$ be a periodic function of period 2π defined on $[0, 2\pi]$ then $f(x)$ can be expressed as

$$f(x) = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + a_3\cos 3x + \cdots + a_n\cos nx + \cdots + b_1\sin x + b_2\sin 2x + b_3\sin 3x + \cdots + b_n\sin nx + \cdots$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx$$

where $a_0, a_1, a_2, \cdots a_n, \cdots, b_1, b_2, \cdots b_n$ are called Fourier constants. (or Fourier coefficients). And can be calculated as

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

Proof. Let

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx \quad \text{--- (1)}$$

Integrating (1) with respect to x from $x = 0$ to $x = 2\pi$, we have

$$\int_0^{2\pi} f(x)\,dx = \frac{1}{2}a_0\int_0^{2\pi} dx + \sum_{n=1}^{\infty} a_n\int_0^{2\pi}\cos nx\,dx + \sum_{n=1}^{\infty} b_n\int_0^{2\pi}\sin nx\,dx$$

$$\Rightarrow \int_0^{2\pi} f(x)\,dx = \frac{1}{2}a_0 \cdot 2\pi + \sum_{n=1}^{\infty} a_n(0) + \sum_{n=1}^{\infty} b_n(0)$$

$$\Rightarrow a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$\left[\because \int_0^{2\pi}\cos nx\,dx = 0 \text{ and } \int_0^{2\pi}\sin nx\,dx = 0.\right]$$

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Again, Integrating (1) after multiplying $\cos nx$ to both sides with respect to x from $x=0$ to $x=2\pi$, we have

$$\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{2}a_0\int_0^{2\pi}\cos nx\,dx + \sum_{n=1}^{\infty} a_n\int_0^{2\pi}\cos^2 nx\,dx$$
$$+\sum_{n=1}^{\infty} b_n\int_0^{2\pi}\sin nx\cos nx\,dx$$

$$\Rightarrow \int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{2}a_0(0)+\sum_{n=1}^{\infty}a_n\int_0^{2\pi}\left(\frac{1-\cos 2nx}{2}\right)dx$$
$$+\sum_{n=1}^{\infty} b_n(0)$$

$$=\sum_{n=1}^{\infty} a_n\cdot\frac{1}{2}\cdot 2\pi$$

Comparing the coefficient of a_n, we have

$$\int_0^{2\pi} f(x)\cos nx\,dx = \pi a_n$$

or, $\boxed{a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx}$

Again, Integrating (1) after multiplying $\sin nx$ to both sides with respect to x from $x=0$ to $x=2\pi$, we have

$$\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{2}a_0\int_0^{2\pi}\sin nx\,dx + \sum_{n=1}^{\infty}a_n\int_0^{2\pi}\cos nx\sin nx\,dx$$
$$+\sum_{n=1}^{\infty}b_n\int_0^{2\pi}\sin^2 nx\,dx$$

$$=\frac{1}{2}a_0(0)+\sum_{n=1}^{\infty}a_n(0)+\sum_{n=1}^{\infty}b_n\cdot\int_0^{2\pi}\frac{1-\cos 2nx}{2}dx$$

$$\int_0^{2\pi} f(x)\sin nx\,dx = \sum_{n=1}^{\infty} b_n\cdot\pi$$

comparing the coefficient of b_n from both sides, we get

$\boxed{b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx}$, Proved.

⑤ Numericals on Fourier Series H-3051

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Domain

Expand

① $f(x) = |x|$, $[-\pi, \pi]$

Deduce $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

We know

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\int_{-\pi}^{0} |x|\,dx + \frac{1}{2\pi}\int_{0}^{\pi} |x|\,dx$$

$$= \frac{1}{2\pi}\left[-\left(\frac{x^2}{2}\right)_{-\pi}^{0}\right] + \frac{1}{2\pi}\left(\frac{x^2}{2}\right)_0^{\pi}$$

$$= \frac{1}{2\pi}\left[0 + \frac{\pi^2}{2}\right] + \frac{1}{2\pi}\left[\frac{\pi^2}{2}\right] = \frac{\pi}{2}$$

Alitér / Note: $\int_{-\pi}^{\pi} f(x)dx = 2\int_0^{\pi} f(x)dx$ if $f(x)$ is even

i.e. $\frac{1}{2\pi}\int_{-\pi}^{\pi} |x|\,dx = \frac{2}{2\pi}\int_0^{\pi} x\,dx = \frac{1}{\pi}\cdot\frac{\pi^2}{2} = \frac{\pi}{2}$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx$$ as $g(x) = |x|\cos nx$

$g(-x) = x\cos nx$

$|x| \geq 0$. $[|-x| = |x|]$

$$a_n = \frac{2}{\pi}\left[x\frac{\sin nx}{n}\Big|_0^{\pi} - \int_0^{\pi} 1.\frac{\sin nx}{n}dx\right]$$

$$a_n = \frac{2}{\pi}\left[\pi\frac{\sin n\pi}{n} + \frac{1}{n^2}\left[\cos nx\right]_0^{\pi}\right]$$

$$= \frac{2}{\pi}\left[0 + \frac{1}{n^2}(\cos n\pi - 1)\right]$$ $[\because \sin n\pi = 0]$

$$a_n = \frac{2}{\pi n^2}\left[(-1)^n - 1\right] =$$

(i) if n is even $a_n = 0$

(ii) if n is odd

$a_n = -\frac{4}{\pi n^2}$ i.e. $a_1 = -\frac{4}{\pi . 1^2}$, $a_3 = -\frac{4}{\pi 3^2}$, ...

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|.\sin nx\,dx = 0$$ as $\int_{-\pi}^{\pi} f(x)dx = 0$ if $f(x)$ is odd

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$\because |x|$

$= f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} a_n \cos nx + 0$

$|x| = \frac{\pi}{2} + \sum_{n=1}^{\infty} \left[-\frac{4}{\pi n^2} \cos nx \right] + 0$
n is odd.

$|x| = \frac{\pi}{2} \quad - \frac{4}{\pi} \left[\frac{1}{1^2} \cos x + \frac{1}{3^2} \cos 3x + \frac{1}{5^2} \cos 5x + \cdots \right]$

Put $x = 0$

$\frac{\pi}{2} = \frac{4}{\pi} \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \right]$

OR, $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

2. Expand
$f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 0, & \text{if } x = 0 \\ 1, & \text{if } 0 < x < \pi \end{cases}$

also deduce $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$

$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{2\pi} \int_{-\pi}^{0} f(x)\, dx + \frac{1}{2\pi} \int_{0}^{\pi} f(x) dx$

$= \frac{1}{2\pi} \int_{-\pi}^{0} (-1)\, dx + \frac{1}{2\pi} \int_{0}^{\pi} 1.\, dx$

$= -\frac{1}{2\pi} \left[x \right]_{-\pi}^{0} + \frac{1}{2\pi} \left[x \right]_{0}^{\pi} = -\frac{1}{2} + \frac{1}{2} = 0$

$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx = \frac{1}{\pi} \int_{-\pi}^{0} -\cos nx\, dx + \frac{1}{\pi} \int_{0}^{\pi} \cos nx\, dx$

$= 0$

$b_n = \frac{1}{\pi} \int_{-\pi}^{0} -\sin nx\, dx + \frac{1}{\pi} \int_{0}^{\pi} \sin nx\, dx$

$= \frac{1}{\pi n} \left[\cos nx \right]_{-\pi}^{0} - \frac{1}{\pi n} \left[\cos nx \right]_{0}^{\pi}$

$= \frac{1}{\pi n} \left[1 - \cos n\pi \right] - \frac{1}{\pi n} \left[\cos n\pi - 1 \right]$

$$b_n = \frac{2}{n\pi}\left[1-\cos n\pi\right]$$

$$= \frac{2}{n\pi}\left[1-(-1)^n\right]$$

$(-1)^n = 1$ if n is even
$(-1)^n = -1$ if n is odd.

$$b_2 = b_4 = 0 = b_6 = \cdots$$

$$b_1 = \frac{4}{\pi},\quad b_3 = \frac{4}{3\pi},\quad b_5 = \frac{4}{5\pi},\ \cdots$$

$$\therefore f(x) = \frac{4}{\pi}\sin x + \frac{4}{3\pi}\sin 3x + \frac{4}{5\pi}\sin 5x \cdots$$

at $x = \frac{\pi}{2}$, $f(x) = 1$

$$1 = \frac{4}{\pi} \quad \frac{4}{3\pi} + \frac{4}{5\pi}$$

$$\boxed{\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots}$$

&

$\sin\frac{3\pi}{2} = \sin\left(\pi+\frac{\pi}{2}\right) = -\sin\frac{\pi}{2}$

$\sin\frac{5\pi}{2} = \sin\frac{\pi}{2}$

$\sin\frac{7\pi}{2} = -1$

$$\left\{\begin{array}{l} \sin\left(\text{odd}\,\pi + \frac{\pi}{2}\right) = -1 \\ \sin\left(\text{even}\,\pi + \frac{\pi}{2}\right) = 1 \end{array}\right.$$

(3) Show that an even fn can have no sine term in its Fourier Series $f(x) = x\sin x$, on $[-\pi, \pi]$, deduct $\frac{\pi}{4} = \frac{1}{2} + \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \frac{1}{7.9} + \cdots$

Sol We have Fourier series as

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\sin x\,dx$$

$$a_0 = \frac{1}{2\pi}2.\int_0^{\pi} x\sin x\,dx \quad \text{as } (x\sin x) \text{ is even fn}$$

$$a_0 = \frac{1}{\pi}\left[-x\cos x\Big|_0^{\pi} + \int_0^{\pi}\cos x\,dx\right]$$

$\left[\because 2\cos A\sin B = \sin(A+B) - \sin(A-B)\right]$

$$a_0 = \frac{1}{\pi}\left[+\pi + 0\right] = 1$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin x\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi}\frac{1}{2}\cdot 2\cos nx\sin x \cdot x\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x.\left[\sin(n+1)x - \sin(n-1)x\right]dx$$

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$$a_n = \frac{1}{\pi}\left[x\left\{-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right\}\Bigg|_0^{\pi} - \int_0^{\pi}\left(\frac{\cos(n-1)x}{(n-1)} - \frac{\cos(n+1)x}{n+1}\right)dx\right.$$

$$a_n = \frac{1}{\pi}\left[\pi\left\{\frac{\cos(n-1)\pi}{n-1} - \frac{\cos(n+1)\pi}{n+1}\right\}\right] - 0$$

$$a_n = \frac{1}{n-1}(-1)^{n-1} - \frac{1}{n+1}(-1)^{n+1}, \quad n \geq 2$$

$$a_2 = \frac{1}{3} - 1, \quad a_3 = \frac{1}{2}(-1)^2 - \frac{1}{4}(-1)^4 = \frac{1}{2} - \frac{1}{4}$$

$$a_4 = \frac{1}{3}(-1)^3 - \frac{1}{5}(-1)^5 = -\frac{1}{3} + \frac{1}{5}$$

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin x \cos x \, dx$$

$$= \frac{\cancel{2}}{\pi}\int_0^{\pi} x \frac{\sin 2x}{\cancel{2}} dx = \frac{1}{\pi}\left[x\left(-\frac{\cos 2x}{2}\right)\Bigg|_0^{\pi} + \int_0^{\pi}\frac{\cos 2x}{2}dx\right]$$

$$= \frac{1}{\cancel{\pi}}\left[\frac{\cancel{\pi}(-1)\cos 2\pi}{2}\right] = -\frac{1}{2}$$

Now,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin x \sin nx \, dx$$

$$\therefore b_n = \frac{1}{\pi} \times 0 = \underline{0}.$$

Let $g(x) = x \sin x \sin nx$

$$g(-x) = -x \sin(-x)\sin(-nx)$$
$$= -x \sin x \sin nx$$
$$= -g(x)$$

i.e $g(x)$ is odd fn

$$\therefore x \sin x = a_0 + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + a_4 \cos 4x + \cdots$$

$$x \sin x = 1 - \frac{1}{2}\cos x - \frac{2}{3}\cos 2x + \frac{1}{4}\cos 3x - \frac{2}{15}\cos 4x \cdots$$

Put $x = \frac{\pi}{2}$

$$\frac{\pi}{2}\sin\frac{\pi}{2} = 1 - \frac{1}{2}\times 0 - \frac{2}{3}\cos\pi + \frac{1}{4}\cos\frac{3\pi}{2} - \frac{2}{15}\cos\frac{4\pi}{2} \cdots$$

$$\frac{\pi}{2} = 1 + \frac{2}{3} - \frac{2}{15} + \cdots \Rightarrow \frac{\pi}{2} = 2\left[\frac{1}{2} + \frac{1}{3} - \frac{1}{15} + \cdots\right]$$

$$\Rightarrow \frac{\pi}{4} = \frac{1}{2} + \frac{1}{1.3} - \frac{1}{3.5} + \cdots$$

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4. Expand $f(x) = \frac{\pi - x}{2}$, $0 < x < 2\pi$ in a Fourier series. Also deduce that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

Soln. $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\pi}{2} - \frac{x}{2}\right)dx$$

$$= \frac{1}{2\cancel{\pi}} \times \frac{\cancel{\pi}}{2}\int_0^{2\pi} dx - \frac{1}{4\pi}\int_0^{2\pi} x\,dx$$

$$= \frac{1}{4}\cdot 2\pi - \frac{1}{\cancel{4}\pi}\cdot\frac{1}{2}\left[\cancel{4}\pi^2\right]$$

$$= \frac{\pi}{2} - \frac{\pi}{2} = 0.$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi}\frac{\pi - x}{2}\cdot\cos nx\,dx$$

$$= \frac{1}{\cancel{\pi}} \times \frac{\cancel{\pi}}{2}\int_0^{2\pi}\cos nx\,dx - \frac{1}{2\pi}\int_0^{2\pi} x\cos nx\,dx$$

$$= \frac{1}{2}\int_0^{2\pi}\cos nx\,dx - \frac{1}{2\pi}\left[x\cdot\frac{\sin nx}{n}\Big|_0^{2\pi} - \int_0^{2\pi}\frac{\sin nx}{n}\,dx\right]$$

$$= 0 - \frac{1}{2\pi}\left[0 - \frac{1}{n}\int_0^{2\pi}\sin nx\,dx\right]$$

$$= \frac{1}{2\pi n}\int_0^{2\pi}\sin nx\,dx = \frac{1}{2\pi n^2}\left[\cos nx\right]_0^{2\pi} = \underline{\underline{0}}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi}\frac{\pi - x}{2}\cdot\sin nx\,dx$$

$$= \frac{1}{2}\int_0^{2\pi}\sin nx\,dx - \frac{1}{2\pi}\int_0^{2\pi} x\sin nx\,dx$$

$$= -\frac{1}{2}\left[\frac{\cos nx}{n}\right]_0^{2\pi} - \frac{1}{2\pi}\left[-\frac{x\cos nx}{n}\Big|_0^{2\pi} + \int_0^{2\pi}\frac{\cos nx}{n}\,dx\right]$$

$$= 0 + \frac{1}{\cancel{2\pi} n}\left[\cancel{2\pi}\cos 2n\pi\right] - \underbrace{\frac{1}{2\pi}\int_0^{2\pi}\frac{\cos nx}{n}\,dx}_{=0} = \frac{1}{n}$$

ii $a_0 = 0$, $a_n = 0$ $b_n = \frac{1}{n}$

ii $\frac{\pi - x}{2} = 0 + 0 + \Sigma \frac{1}{n} \sin nx$

$\frac{\pi - x}{2} = \sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \frac{1}{4}\sin 4x + \cdots$

Put $x = \frac{\pi}{2}$

$\frac{\pi}{4} = 1 + 0 + \frac{1}{3}(-1) + \frac{1}{4}(0) + \frac{1}{5}1 + \frac{1}{6}\times 0$
$+ \frac{1}{7}(-1) + \cdots$

$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$

(5) obtain the Fourier Series of $f(x) = x^2$ in $[-\pi, \pi]$ and Show that

(i) $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots$

(ii) $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \cdots$

(iii) $\frac{\pi^2}{48} = \frac{1}{2^2} - \frac{1}{4^2} + \frac{1}{6^2} - \cdots$

(iv) $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \cdots$

Sol we have F. series as

$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\cancel{2}\pi}\cancel{2}.\int_0^{\pi} x^2 dx = \frac{1}{3\pi}\pi^3 = \frac{\pi^2}{3}$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,dx$

$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right)\Big|_0^{\pi} - \int_0^{\pi} 2x \frac{\sin nx}{n} dx \right]$

$= -\frac{4}{\pi n}\int_0^{\pi} x \sin nx\,dx$

$$a_n = -\frac{4}{\pi n}\int_0^{\pi} x \sin nx\, dx$$

$$= -\frac{4}{\pi n}\left[-x\frac{\cos nx}{n}\Big|_0^{\pi} + \int_0^{\pi}\frac{\cos nx}{n}\,dx\right]$$

$$= \frac{4}{\pi n^2}\left[x\cos nx\right]_0^{\pi} - \underbrace{\frac{4}{\pi n^2}\int_0^{\pi}\cos nx\,dx}_{=0}$$

$$= \frac{4}{\pi n^2}\left[\pi\cos n\pi\right]$$

$$a_n = \frac{4}{\cancel{\pi} n^2}\left[\cancel{\pi}(-1)^n\right] = \frac{4}{n^2}(-1)^n$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin nx\, dx = 0 \text{ as } (x^2\sin nx) \text{ is odd fn}$$

$$\therefore\ x^2 = a_0 + \sum a_n \cos nx + \sum b_n \sin nx$$

$$= \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n\cos nx + 0$$

$$x^2 = \frac{\pi^2}{3} - \frac{4\cos x}{1^2} + \frac{4\cos 2x}{2^2} - \frac{4}{3^2}\cos 3x + \frac{4}{4^2}\cos 4x - -$$

$$x^2 - \frac{\pi^2}{3} = -4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + --\right]$$

(i) Put $x = \pi$

$$\pi^2 - \frac{\pi^2}{3} = -4\left[\frac{\cos\pi}{1^2} - \frac{\cos 2\pi}{2^2} + \frac{\cos 3\pi}{3^2} - \frac{\cos 4\pi}{4^2} + --\right]$$

$$\frac{2\pi^2}{3} = -4\left[-\frac{1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} - --\right]$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\cdots$$

(ii) Put $x = 0$

$$-\frac{\pi^2}{3} = -4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + --\right]$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

(ii) Put $x = \frac{\pi}{2}$

$$\frac{\pi^2}{4} - \frac{\pi^2}{3} = -4\left[\frac{1}{1^2}\cos\frac{\pi}{2} - \frac{1}{2^2}\cos 2\cdot\frac{\pi}{2} + \frac{1}{3^2}\cos 3\frac{\pi}{2} - \frac{1}{4^2}\cos 4\cdot\frac{\pi}{2} + \cdots\right]$$

$$\frac{-\pi^2}{12} = -4\left[0 + \frac{1}{2^2} - \frac{1}{3^2}\times 0 - \frac{1}{4^2} + \cdots\right]$$

$$\frac{\pi^2}{48} = \frac{1}{2^2} - \frac{1}{4^2} + \frac{1}{6^2} - \cdots$$

(iii) Adding (i) and (ii), we get

$$\frac{\pi^2}{6} + \frac{\pi^2}{12} = \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right) + \left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\right)$$

$$\frac{6\pi^2}{24} = 2\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

6 Find Fourier Series of

$$f(x) = \begin{cases} x-\pi, & -\pi < x < 0 \\ \pi - x, & 0 < x < \pi \end{cases}$$

We have

$$f(x) = a_0 + \sum a_n \cos nx + \sum b_n \sin nx$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{0} f(x)dx + \frac{1}{2\pi}\int_0^{\pi} f(x)dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{0}(x-\pi)dx + \frac{1}{2\pi}\int_0^{\pi}(\pi - x)dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{0} x\,dx - \frac{1}{2}\int_{-\pi}^{0} dx + \frac{1}{2}\int_0^{\pi} dx - \frac{1}{2\pi}\int_0^{\pi} x\,dx$$

$$= \frac{1}{2\pi}\cdot\frac{1}{2}\left[x^2\right]_{-\pi}^{0} - \frac{1}{2}\left[x\right]_{-\pi}^{0} + \frac{1}{2}\left[x\right]_0^{\pi} - \frac{1}{2\pi}\cdot\frac{1}{2}\left[x^2\right]_0^{\pi}$$

$$= \frac{1}{4\pi}\left[0-\pi^2\right] - \frac{1}{2}\left[0+\pi\right] + \frac{1}{2}\left[\pi\right] - \frac{1}{4\pi}\left[\pi^2\right]$$

$$= -\frac{\pi}{4} - \cancel{\frac{\pi}{2}} + \cancel{\frac{\pi}{2}} - \frac{\pi}{4} = -\frac{2\pi}{4} = -\frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0}(x-\pi)\cos nx\,dx + \frac{1}{\pi}\int_0^{\pi}(\pi - x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} x\cos nx\,dx - \int_{-\pi}^{0}\cos nx\,dx + \int_0^{\pi}\cos nx\,dx - \frac{1}{\pi}\int_0^{\pi} x\cos nx\,dx$$

$$= \frac{1}{\pi}\left[x\frac{\sin nx}{n}\Big|_{-\pi}^{0} - \int_{-\pi}^{0}\frac{\sin nx}{n}dx\right] - 0 + 0 - \frac{1}{\pi}\left[x\frac{\sin nx}{n}\Big|_0^{\pi} - \int_0^{\pi}\frac{\sin nx}{n}dx\right]$$

$$= \frac{1}{\pi}\left[-\frac{1}{n}\int_{-\pi}^{0}\sin nx\,dx\right] - \frac{1}{\pi n}\left[-\int_0^{\pi}\sin nx\,dx\right]$$

$$= +\frac{1}{\pi n^2}\left[\cos nx\right]_{-\pi}^{0} - \frac{1}{\pi n^2}\left[\cos nx\right]_0^{\pi} = \frac{1}{\pi n^2}\left[1-(-1)^n - (-1)^n + 1\right]$$

$$a_n = \frac{2}{\pi n^2}\left[1-(-1)^n\right]$$

$$a_n = \frac{2}{\pi n^2}\left[1+(-1)^{n+1}\right]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0}(x-\pi)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi}(\pi-x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} x\sin nx\,dx - \int_{-\pi}^{0}\sin nx\,dx + \int_{0}^{\pi}\sin nx\,dx - \frac{1}{\pi}\int_{0}^{\pi} x\sin nx\,dx$$

$$= \frac{1}{\pi}\left[-\frac{x\cos nx}{n}\Big|_{-\pi}^{0} + \int_{-\pi}^{0}\frac{\cos nx}{n}\,dx\right] + \frac{1}{n}(\cos nx)_{-\pi}^{0} - \frac{1}{n}\left[\cos nx\right]_{0}^{\pi}$$

$$- \frac{1}{\pi}\left[-\frac{x\cos nx}{n}\Big|_{0}^{\pi} + \int_{0}^{\pi}\frac{\cos nx\,dx}{n} = 0\right]$$

$$= \frac{1}{\pi}\left[0 - \frac{\pi\cos\pi n}{n} + \frac{1}{n}(0)\right] + \frac{1}{n}\left[1-(-1)^n\right] - \frac{1}{n}\left[(-1)^n - 1\right]$$

$$+ \frac{1}{\pi n}\left[\pi\cos n\pi\right] - \frac{1}{\pi}\times 0$$

$$= \cancel{-\frac{1}{n}(-1)^n} + \frac{1}{n} \cancel{-\frac{1}{n}(-1)^n} - \frac{1}{n}(-1)^n + \frac{1}{n} + \frac{1}{n}(-1)^n$$

$$= \frac{2}{n} - \frac{2}{n}(-1)^n$$

$$b_n = \frac{2}{n}\left[1-(-1)^n\right] = \frac{2}{n}\left[1+(-1)^{n+1}\right]$$

$$\therefore f(x) = -\frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2}{\pi}\left[\frac{1+(-1)^{n+1}}{n^2}\right]\cos nx + \sum_{n=1}^{\infty}\frac{2}{n}\left[1+(-1)^{n+1}\right]\sin nx$$